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ON THE APPLICATION OF INTEGRAL-METHODS TO THE SOLUTION OF PROBLEMS INVOLVING THE SOLIDIFICATION OF LIQUIDS INITIALLY AT FUSION TEMPERATURE

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Abstract—Approximate integral-methods, similar to those used in the solution of the boundary layer equations in fluid dynamics, are developed for determining the location and time history of the uni-dimensional solid–liquid interface during the solidification of liquids initially at the fusion temperature. For isothermal wall condition the configurations treated are the inward solidification of the semi-infinite region, the circular cylinder and the sphere. Series solutions of the thermal equation giving the location of the solid–liquid interface, for small time, are derived for the latter two shapes and are used to estimate the accuracy of the approximate integral-methods. The approximate results for the circular cylinder are also compared with an existing numerical solution obtained by the relaxation method.

NOMENCLATURE

a, representative length;
c, specific heat;
k, thermal diffusivity;
L, latent heat of fusion;
t, time;
T, temperature;
β, $\dfrac{L}{c(T_F - T_0)}$, dimensionless group;
λ, configuration parameter, i.e. $\lambda = 0$, 1 and 2 for the semi-infinite region, circular cylinder and sphere respectively;
τ, $\dfrac{tk}{a^2}$, dimensionless time;
κ, thermal conductivity;
Θ, $(T - T_0)/(T_F - T_0)$, dimensionless temperature;
Θ^*, dimensionless energy thickness;
ϵ, dimensionless length representing the depth of solidification.

Subscripts
0, wall condition;
F, solid–liquid interface condition.

1. INTRODUCTION

WHEN a liquid is undergoing a change of phase from liquid to solid, thermal energy in the form of the latent heat of fusion is released at a moving solid–liquid interface whose location and time history is unknown. Since the boundary conditions at the solid–liquid interface are non-linear in the temperature it may be expected that analytical solutions will be difficult to obtain. One of the few exact analytical solutions (see Carslaw and Jaeger [1] is that found by Neumann and Stefan relating to the solidification of the semi-infinite region of liquid initially above or at the fusion temperature when the surface wall temperature is suddenly decreased to or maintained at a temperature below fusion. Further work has been done on this configuration by Evans *et al.* [2] who derived, using series expansions, the position of the solidification front for small time, when a given heat flux was prescribed at the wall. When the wall temperature is adjusted so that there is a constant rate of solidification, the boundary conditions at the solid–liquid interface are then linear, and so analytical solutions may be obtained. This type of problem was first investigated by Stefan (see Ingersoll *et al.* [3]) for the semi-infinite region and by Kreith and Romie [4] for the inward solidification of the circular cylinder and sphere initially at the fusion temperature.

With regard to the non-linear problem in which the position of the solidification front is unknown, solutions may be obtained by numerical methods. Landau [5], using a finite difference step-by-step procedure has presented a set of solutions for the semi-infinite region (see also Eyres *et al.* [6]); the convergence and stability of this procedure has been investigated by Douglas and Gallie [7]. Another approach to the solution of the solidification problem involving a plane front is given by Crank [8]. Here the numerical solution of the diffusion equation with a moving boundary is replaced, on making an appropriate change of variable, by an equivalent eigenvalue type problem with fixed boundaries. Finally in [9] and [10] Allen and Severn have treated, using the method of relaxation, the unidimensional problems of the semi-infinite region and the cylinder respectively.

In the present paper approximate integral methods, similar to those used in the solution of boundary layer problems in fluid dynamics, are developed and applied to problems involving unidimensional solidification fronts.

2. THE MATHEMATICAL PROBLEM

In the following discussion the liquid is assumed to satisfy the conditions: (i) the liquid has a definite fusion temperature, (ii) initially the liquid is at fusion temperature and (iii) all thermal properties of the material are uniform and constant. Thus the temperature distribution in the solidified phase is given by the thermal diffusion equation

$$k\nabla^2 T = \frac{\partial T}{\partial t}. \tag{2.1}$$

At the solid–liquid interface $T = T_F$, where T_F is the fusion temperature, and at the wall surface $T = T_0 < T_F$. The initial conditions are $T = T_F$ at $t = 0$, and no solidification has occurred.

Consider now the boundary conditions associated with the unidimensional solidification of the semi-infinite region, the cylinder and the sphere. If $E(t)$ denotes the depth of penetration of the front then for the semi-infinite region, when $t > 0$

$T = T_0$ at $x = 0$, $T = T_F$ and

$$\kappa \frac{\partial T}{\partial x} = \rho L \frac{dE}{dt} \text{ at } x = E, \tag{2.2a}$$

where x is measured in a normal inward direction from the plane surface: for the circular cylinder or sphere, when $t > 0$

$T = T_0$ at $r = a$, $T = T_F$ and

$$\kappa \frac{\partial T}{\partial r} = -\rho L \frac{dE}{dt} \text{ at } r = a - E, \tag{2.2b}$$

where a is the radius of the cylinder or the sphere; also at $t = 0$

$$E = 0 \text{ and } T = T_F. \tag{2.3}$$

On examination of the boundary conditions at the solid–liquid interface it follows that the latter two conditions of (2.2a) and (2.2b) are equivalent to

$$\frac{\partial T}{\partial t} = -\frac{\kappa}{\rho L}\left(\frac{\partial T}{\partial x}\right)^2 \text{ at } x = E \text{ or}$$

$$\frac{\partial T}{\partial t} = -\frac{\kappa}{\rho L}\left(\frac{\partial T}{\partial r}\right)^2 \text{ at } r = a - E \tag{2.4}$$

respectively. Thus the equations (2.1) to (2.4) are similar to the boundary layer equations in fluid dynamics and E could be identified with the boundary layer thickness. In both cases the equations to be solved are parabolic and non-linear.

Following the methods already developed for obtaining exact solutions of the boundary layer equations we shall now obtain a power series representation for $E(t)$ about $t = 0$. We introduce the dimensionless modulii and new variables

$$\epsilon = \frac{E}{a}, \ \Theta(\eta, \tau) = \frac{(T - T_0)}{(T_F - T_0)},$$

$$\beta = \frac{L}{c(T_F - T_0)}, \ \eta = \frac{(a - r)}{a\epsilon} \text{ and } \tau = \frac{tk}{a^2}. \tag{2.5}$$

The basic equation (2.1) becomes

$$\left.\begin{aligned} &\frac{\partial^2\Theta}{\partial\eta^2} + \left(\epsilon\frac{d\epsilon}{d\tau}\right)\eta\frac{\partial\Theta}{\partial\eta} - \epsilon^2\frac{\partial\Theta}{\partial\tau} = \frac{\lambda\epsilon}{(1 - \epsilon\eta)}\frac{\partial\Theta}{\partial\eta} \\ &\text{subject to the boundary conditions:} \\ &\Theta = 0 \text{ at } \eta = 0, \ \Theta = 1 \text{ and} \\ &\frac{\partial\Theta}{\partial\eta} = \beta\epsilon\frac{d\epsilon}{d\tau} \text{ at } \eta = 1. \end{aligned}\right\} \tag{2.6}$$

In equation (2.6) $\lambda = 1$ for the circular cylinder, and $\lambda = 2$ for the sphere. The case $\lambda = 0$ corresponds to the inward solidification of a region bounded by the plane wall $x = 0$ and a thermally insulated wall at $x = a$, where $(\partial T/\partial x)_{x=a} = 0$. The solution of equations (2.6) for $\lambda = 0$, where now $x = (a - r)$ and r is measured from the insulated wall, will then apply for the period of solidification. The complication of non-linearity is reduced by expressing ϵ and Θ as series in powers of τ; i.e.

$$\epsilon = \sum_{r=0}^{\infty} \epsilon_r \, \tau^{1/2(1+r)}, \; \Theta = \sum_{r=0}^{\infty} f_r(\eta)\tau^r. \qquad (2.7)$$

On substituting the series (2.7) into (2.6) and equating coefficients of like powers of τ, there results a set of equations of which the first two are:

$$\left.\begin{aligned} & f_0'' + 2\zeta f_0' = 0, \; f_0(0) = 0, \\ & \quad f_0\left(\frac{\epsilon_0}{2}\right) = 1, \; f_0'\left(\frac{\epsilon_0}{2}\right) = \beta\epsilon_0; \\ & f_1'' + 2\zeta f_1' - 2f_1 = \frac{\epsilon_0}{2}(\lambda\epsilon_0 - 3\epsilon_1\zeta)f_0', \\ & \quad f_1(0) = f_1\left(\frac{\epsilon_0}{2}\right) = 0, f_1'\left(\frac{\epsilon_0}{2}\right) = 3\beta\epsilon_1. \end{aligned}\right\} (2.8)$$

Here the primes denote differentiation with respect to the new variable $\zeta = \frac{1}{2}\epsilon_0\eta$. Using the two point boundary conditions on $f_r(\zeta)$ the equations (2.8) may be solved exactly. The gradient condition on $f_r(\zeta)$ then determines the unknown constants ϵ_r. The method of solution is not given here as this has already been described by Goldstein and Rosenhead [11]. The first three coefficients ϵ_0, ϵ_1 and ϵ_2 may be found from the following relations:

$$\beta\epsilon_0 \, e^{\epsilon_0^2/4} \int_0^{\epsilon_0/2} e^{-t^2} \, dt = 1, \quad \epsilon_1 = \frac{\lambda\epsilon_0^2}{6 + \epsilon_0^2}$$

and
$$\epsilon_2 = \frac{-\epsilon_0}{\beta\left\{3 + \frac{1}{2}\epsilon_0^2 + \frac{2(1+\beta)\epsilon_0^2}{2(1+\beta)\epsilon_0^2}\right\}}$$

$$\left[\lambda + \beta^2\left\{\tfrac{1}{4}\lambda(2 + 3\lambda) - \tfrac{1}{2}\lambda\epsilon_1 + \tfrac{3}{4}\left(1 + \frac{4}{\epsilon_0^2}\right)\epsilon_1^2 - \epsilon_0^2(\tfrac{1}{8}\lambda(2 + \lambda) - \tfrac{1}{4}\lambda\epsilon_1 + \tfrac{1}{8}\epsilon_1^2)\right\}\right.$$

$$\left. - \frac{(1+\beta)}{2 + (1+\beta)\epsilon_0^2}\left\{\lambda^2(2 + \epsilon_0^2) + \tfrac{1}{2}\beta\left[\epsilon_1^2(4 + \epsilon_0^2) + \epsilon_0^2\lambda(\lambda + 2)\right]\right\}\right] \qquad (2.9)$$

Further coefficients ϵ_r may be evaluated but are tedious to obtain if $r > 2$. Needless to say the series (2.7) have limited usefulness and moreover the radius of convergence would be difficult to obtain. The same difficulty exists in the Blasius-type expansions of the boundary layer equations* (see Goldstein [12]) and for this reason it is desirable to obtain information on the solidification process by using the approximate integral-methods of boundary layer theory.

3. APPROXIMATE ANALYSIS FOR THE TREATMENT OF UNIDIMENSIONAL SOLIDIFICATION FRONTS

Following the approximate methods used in boundary layer theory we shall abandon the attempt to satisfy the heat conduction equation at every point and time and instead assume a temperature profile which is made to satisfy certain integrated forms of the heat conduction equation for the solidified phase. For convenience we introduce the dimensionless moduli

$$\epsilon = \frac{E}{a}, \quad \Theta(X, \tau) = \frac{(T - T_0)}{(T_F - T_0)},$$
$$\beta = \frac{L}{c(T_F - T_0)}, \quad \tau = \frac{tk}{a^2} \qquad (3.1)$$

$X = \dfrac{x}{a}$ for $\lambda = 0$ and $X = \dfrac{(a - r)}{a}$ for $\lambda = 1$ or 2.

Equations (2.1) to (2.4) now become

$$\frac{\partial}{\partial X}\left\{(1 - X)^\lambda \frac{\partial\Theta}{\partial X}\right\} = (1 - X)^\lambda \frac{\partial\Theta}{\partial\tau}, \qquad (3.2a)$$

$$\Theta(0, \tau) = 0, \quad \Theta(\epsilon, \tau) = 1, \qquad (3.2b)$$

$$\Theta_X(\epsilon, \tau) = \beta \frac{d\epsilon}{d\tau}. \qquad (3.2c)$$

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* Note that the inward solidification of the cylinder is analogous to the development of the laminar boundary layer in the inlet length of a circular pipe (see Goldstein and Atkinson [12]).

$$\{\Theta_X(\epsilon,\tau)\}^2 = -\frac{1}{\beta}\Theta_\tau(\epsilon,\tau), \qquad (3.2d)$$

and $$\Theta(X, 0) = 1, \quad \epsilon(0) = 0, \qquad (3.2e)$$

where the suffixes denote first order partial differentiation. The heat balance integral for the solidified phase is obtained by integrating both sides of equation (3.2a) from $X = 0$ to $X = \epsilon$ and applying the boundary condition (3.2c). The result is

$$\beta(1-\epsilon)^\lambda \frac{d\epsilon}{d\tau} - \left(\frac{\partial\Theta}{\partial X}\right)_{X=0} = \int_0^\epsilon (1-X)^\lambda \frac{\partial\Theta}{\partial\tau}\, dX. \qquad (3.3)$$

In a similar fashion by multiplying both sides of equation (3.2a) by $(1-X)^\lambda\,(\partial\Theta/\partial X)\,dX$, integrating from $X = 0$ to $X = \epsilon$ and applying the boundary condition (3.2d), we obtain

$$\beta(1-\epsilon)^{2\lambda}\left(\frac{\partial\Theta}{\partial\tau}\right)_{X=\epsilon} + \left(\frac{\partial\Theta}{\partial X}\right)^2_{X=0} = -2\int_0^\epsilon (1-X)^{2\lambda}\frac{\partial\Theta}{\partial X}\frac{\partial\Theta}{\partial\tau}\, dX. \qquad (3.4)$$

Equation (3.4) has less physical significance than equation (3.3), but may be used in conjunction with (3.3) to obtain approximate information for ϵ. Two different approximations are developed.

I—*The Kármán–Pohlhausen method*

We assume the simple one-parameter temperature profile

$$\Theta = \frac{X}{\epsilon}, \qquad (3.5)$$

which satisfies the conditions (3.2b). The unknown parameter, i.e. the dimensionless thickness of solidified material $\epsilon(\tau)$, is then obtained on direct substitution of (3.5) into the heat balance integral (3.3). The solutions are: for the semi-infinite region, $\lambda = 0$,

$$\tau = \tfrac{1}{4}(2\beta+1)\epsilon^2 = \left(\frac{\epsilon}{\epsilon_0}\right)^2; \qquad (3.6)$$

for the circular cylinder, $\lambda = 1$,

$$\tau = \tfrac{1}{4}(2\beta+1)\epsilon^2 - \tfrac{1}{9}(3\beta+1)\epsilon^3; \qquad (3.7)$$

and for the sphere, $\lambda = 2$,

$$\tau = \tfrac{1}{4}(2\beta+1)\epsilon^2 - \tfrac{2}{9}(3\beta+1)\epsilon^3 + \tfrac{1}{16}(4\beta+1)\epsilon^4. \qquad (3.8)$$

This method is that originally due to Kármán and Pohlhausen (see Goldstein [12]) for solving the momentum integral equation of fluid dynamics. The Kármán–Pohlhausen technique has been applied by Goodman [13] to investigate the unidimensional solidification of the semi-infinite region. Here a one-parameter quadratic profile was chosen to satisfy the conditions (3.2b) and the non-linear boundary condition (3.2d). Unfortunately this method appears to fail when applied to the circular cylinder and the sphere. Thus the second method to be used in this paper is based on a modification of the Kármán–Pohlhausen technique due to Tani [14].

II—*The Tani method*

The essential idea of the application of the Tani method to problems involving unidimensional fronts is to assume a simple two-parameter quadratic profile which satisfies the conditions (3.2b), i.e.

$$\Theta = \frac{X}{\epsilon} + g\left(\frac{X}{\epsilon} - \frac{X^2}{\epsilon^2}\right). \qquad (3.9)$$

The two unknown parameters $\epsilon(\tau)$ and $g(\tau)$ are then obtained on solving the pair of first-order equations derived on substitution of (3.9) into the integral equations (3.3) and (3.4). The initial conditions are:

$$\epsilon(0) = 0 \text{ and } \lim_{\epsilon\to 0} \epsilon g = 0. \qquad (3.10)$$

The latter condition is derived from consideration of the total thermal energy of the solidified phase. This is related to the energy thickness

$$\Theta^*(\epsilon,\lambda) = \textstyle\int_0^\epsilon (1-X)^\lambda\,\Theta\, dX. \qquad (3.11)$$

and on using expression (3.9) the above condition on g must be satisfied since $\Theta^*(0,\lambda) \equiv 0$.

Consider first the semi-infinite region. The resulting equations for $\epsilon(\tau)$ and $g(\tau)$ are:

$$\tau = -\tfrac{1}{6}\,\epsilon^2 \log(1+g) + \int_0^{\epsilon}\left\{\tfrac{1}{3}\log(1+g) + \frac{(6\beta+3-g)}{6(1+g)}\right\}\epsilon\, d\epsilon \qquad (3.12)$$

and

$$(1-g)\epsilon\frac{dg}{d\epsilon} = 3\,\{1 - 2(1+\beta)g + g^2\},$$
$$\lim_{\epsilon\to 0}\ \epsilon g = 0. \qquad (3.13)$$

Equation (3.13) is satisfied identically by

$$g = g(0) = 1 + 2\beta - 2\surd(\beta+\beta^2), \qquad (3.14)$$

a constant. The equation for $\tau(\epsilon)$ can be integrated exactly giving

$$\tau = \frac{6\beta + 3 - g(0)}{12\{1+g(0)\}}\,\epsilon^2 = \left(\frac{\epsilon}{\epsilon_0}\right)^2, \qquad (3.15)$$

where on using (3.14)

$$\epsilon_0^2 = \frac{12\{1+\beta-\surd(\beta+\beta^2)\}}{1+2\beta+\surd(\beta+\beta^2)}. \qquad (3.16)$$

The determination of $\tau(\epsilon)$ and $g(\epsilon)$ in the case of the circular cylinder ($\lambda = 1$) or the sphere ($\lambda = 2$) is a little more complicated since for these configurations g is no longer constant. Equations will only be given for the circular cylinder, namely

$$\tau = -\tfrac{1}{12}\,\epsilon^2(2-\epsilon)\log(1+g) + \tfrac{1}{12}\int_0^{\epsilon}\left\{(4-3\epsilon)\log(1+g) + \frac{12\beta + 6 - (12\beta+4)\epsilon + 2(\epsilon-1)g}{(1+g)}\right\}\epsilon\, d\epsilon, \qquad (3.18)$$

and

$$\{(10 - 15\epsilon + 6\epsilon^2) - (10 - 9\epsilon + 2\epsilon^2)g\} \times \epsilon\frac{dg}{d\epsilon} = \{30 - 60(1+\beta)\epsilon + 30(1+\beta)\epsilon^2\}$$
$$- g\{60(1+2\beta) - 90(1+2\beta)\epsilon + 12(3+5\beta)\epsilon^2\} + g^2\{30 - 42\epsilon + 14\epsilon^2\},$$
$$\lim_{\epsilon\to 0}\ \epsilon g = 0. \qquad (3.19)$$

Equation (3.19) cannot be integrated exactly but may be quickly solved numerically by the method of Fox and Goodwin [15] for the range $0 \leqslant \epsilon \leqslant 1$. The required initial values to be used in this method are:

$$g(0) = 1 + 2\beta - 2\surd(\beta+\beta^2) \text{ and } g'(0) = -\left\{\frac{30(1+\beta) - 45(1+2\beta)g(0) + 21g^2(0)}{35 - 35g(0) + 60\beta}\right\}. \qquad (3.19a)$$

These have been obtained using a series expansion for $g(\epsilon)$ at the origin, $\epsilon = 0$. Once g has been obtained $\tau(\epsilon)$ may be found from expression (3.18) by direct numerical integration. We note from expression (3.18) that $g > -1$ for $0 \leqslant \epsilon \leqslant 1$ otherwise the approximate method would fail.

4. RESULTS AND DISCUSSION

Results are given graphically in Fig. 1 for the semi-infinite region. The constant ϵ_0 as evaluated from the approximate expressions (3.6) and (3.16) is compared with the exact coefficient evaluated from the transcendental equation in (2.9). It is seen that the Kármán–Pohlhausen

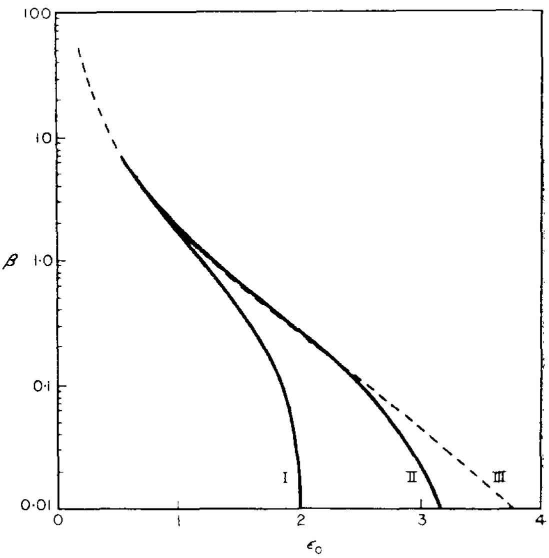

FIG. 1. Curves for obtaining the depth of solidification of a semi-infinite region: I—Kármán–Pohlhausen method, II—Tani method and III—the exact solution.

one-parameter method is sufficient for all practical purposes if $\beta \geq 1$, whilst the Tani two-parameter method is satisfactory for $\beta \geq 1/10$. Note that as $\beta \to \infty$, i.e. the heat capacity of the material becomes vanishingly small, then both approximations give the correct asymptotic form $\epsilon_0 = \sqrt{(2/\beta)}$. It is interesting to compare the numerical magnitude of ϵ_0, for a particular value of β, found using the above approximations with that given by Goodman [13] and that obtained by Allen and Severn [9] using the relaxation method. Taking $\beta = 1{\cdot}5613$, as in [8], the exact result is $\epsilon_0 = 1{\cdot}034$: Allen and Severn give $\epsilon_0 = 1{\cdot}03_6$; the methods of the present paper give $\epsilon_0 = 0{\cdot}985$ and $\epsilon_0 = 1{\cdot}049$ for the Kármán–Pohlhausen and Tani methods respectively; the Goodman procedure using a one-parameter Kármán–Pohlhausen method and assuming a quadratic profile satisfying the boundary conditions (3.2b) and (3.2d) gives $\epsilon_0 = 1{\cdot}089_5$.

In Fig. 2 the result obtained by Allen and Severn [10] for the inward solidification of the circular cylinder is reproduced ($\beta = 1{\cdot}5613$). The agreement between their result and that obtained by the approximate methods of section 3 is seen to be satisfactory. In particular the non-dimensional time for complete solidification is $\tau = 0{\cdot}40$ and $\tau = 0{\cdot}52$ for methods I and II respectively as compared with the result of the relaxation solution, $\tau = 0{\cdot}47$. Moreover for small depths of solidification methods I and II predict nearly identical solidification rates which are in good agreement with the exact values obtained from expressions (2.7) and (2.9). For example in the case of the circular cylinder the exact result for small time is

$$\epsilon = 1{\cdot}034\tau^{1/2} + 0{\cdot}151\tau + 0{\cdot}083\tau^{3/2} + \ldots \quad (4.1)$$

and that obtained by method II is

$$\epsilon = 1{\cdot}049\tau^{1/2} + 0{\cdot}168\tau + 0{\cdot}073\tau^{3/2} + \ldots \quad (4.2)$$

Similar trends were established in the case of the inward solidification of the sphere. For $\beta = 1{\cdot}5613$, the exact result for small time is

$$\epsilon = 1{\cdot}034\tau^{1/2} + 0{\cdot}302\tau + 0{\cdot}198\tau^{3/2} + \ldots \quad (4.3)$$

and that obtained by method II is

$$\epsilon = 1{\cdot}049\tau^{1/2} + 0{\cdot}336\tau + 0{\cdot}160\tau^{3/2} + \ldots \quad (4.4)$$

The complete results for methods I and II are displayed in Fig. 3. Note that the points indicated on the curves in Figs. 2 and 3 have been derived using the expansions (4.1) and (4.3) respectively.

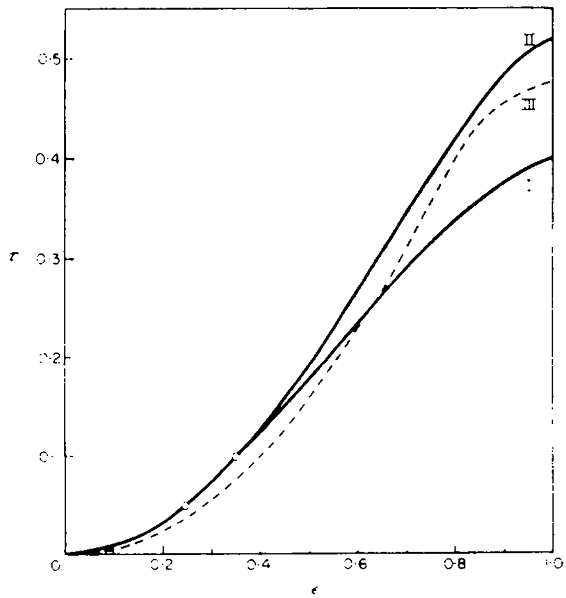

FIG. 2. Depth of solidification and time history for the circular cylinder, when $\beta = 1{\cdot}5613$: I—Kármán–Pohlhausen method, II—Tani method and III—Allen and Severn [10].

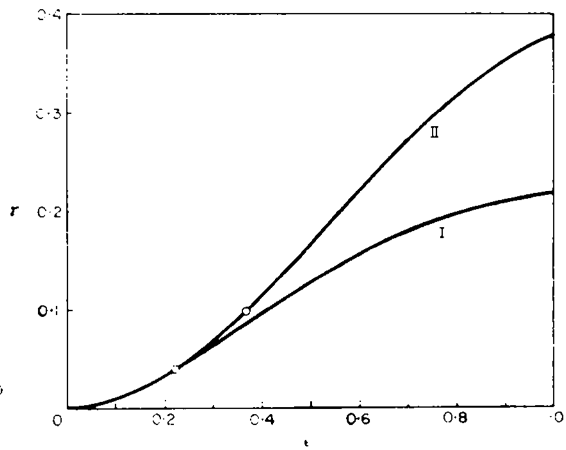

FIG. 3. Depth of solidification and time history for the sphere, when $\beta = 1{\cdot}5613$: I—Kármán–Pohlhausen method and II—Tani method.

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Résumé—L'auteur étudie des méthodes intégrales approchées, semblables à celles utilisées pour résoudre les équations de la couche limite en dynamique des fluides, pour déterminer la position et l'évolution dans le temps de la surface de séparation unidimensionnelle solide–liquide, au cours de la solidification de liquides initialement à la température de fusion. Dans le cas d'une paroi isotherme, les configurations étudiées sont la solidification à l'intérieur d'une région semi-infinie, d'un cylindre circulaire et d'une sphère. Pour ces deux dernières formes, l'auteur a trouvé des solutions (séries) de l'équation de la chaleur qui donnent la position de l'interface solide–liquide pour un temps bref, et qui permettent d'évaluer la précision des méthodes intégrales approchées. Les résultats pour le cylindre circulaire sont également comparés à une solution numérique obtenue par la méthode de relaxation.

Zusammenfassung—Die Orts- und Zeitabhängigkeit der eindimensionalen Verfestigungsfront in einer Flüssigkeit von Schmelztemperatur kann nach angenäherten Integralmethoden bestimmt werden. Dieses Verfahren gleicht dem der Hydrodynamik für die Lösung von Grenzschichtgleichungen. Für den halbunendlichen Bereich, den Zylinder und die Kugel wurde die ins Innere fortschreitende Verfestigung bei isothermen Wandbedingungen untersucht. Die Ortsabhängigkeit der Verfestigungsfront bei kleinen Zeiten, lässt sich für die beiden letzteren Formen in Reihenlösungen angeben. Diese dienten auch zur Abschätzung der Genauigkeit der angenäherten Integralmethoden. Darüber hinaus wurden die Ergebnisse für den Zylinder noch mit einer numerischen Lösung nach Relaxationsmethode verglichen.

Аннотация—Приближенные интегральные методы, подобные используемым в гидродинамике для решения уравнений пограничного слоя, развиты для определения изменения местоположения одномерной поверхности раздела твердой и жидкой фазы при затвердевании жидкостей,находящихся вначале при температуре плавления. При условиях изотермичности на границе рассматривается внутреннее затвердевание полубесконечной области, круглого цилиндра и шара. Для последних двух конфигураций получен ряд решений уравнения теплопроводности, которые позволяют для малых времен получить положение поверхности раздела твердой и жидкой фазы. Эти решения использованы для оценки точности приближенных интегральных методов. Даётся сравнение приближенных результатов для круглого цилиндра с имеющимся численным решением, полученным методом релаксации.